

STELLAR THEORY FOR FLAG COMPLEXES

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ABSTRACT. We show that two *flag* simplicial complexes are piecewise linearly homeomorphic if and only if they can be connected by a sequence of flag complexes, each obtained from the previous one by either an edge subdivision or its inverse. We apply this result to flag spheres and pose new conjectures on their combinatorial structure forced by their face numbers, analogues to the extremal examples in the upper and lower bound theorems for simplicial spheres.

1. INTRODUCTION

A basic result in piecewise linear (PL-)topology, is that

Theorem 1.1. (*Alexander*, [2, Theorem 15:1]) *Two simplicial complexes are PL-homeomorphic if and only if they can be connected by a sequence of stellar subdivisions and their inverses.*

See e.g. [11, Theorem 4.5] for a modern proof and further references.

A (abstract) simplicial complex is called *flag* if all its minimal non-faces have size two; equivalently, it is the complex of cliques of a simple graph. Flag complexes arise in many mathematical contexts, and often interesting families of flag complexes share the same PL-type; for example, the order complexes of intervals w.r.t. Bruhat order on Coxeter groups are PL-spheres [5].

Our main result says that

Main Theorem 5.1. *Two flag simplicial complexes are PL-homeomorphic if and only if they can be connected by a sequence of flag complexes, each obtained from the previous one by either an edge subdivision or its inverse.*

Equivalently, in graph language, this theorem reads as:

Corollary 1.2. *The clique complexes of graphs G and G' are PL-homeomorphic iff there is a sequence of graphs $G = G_0, G_1, \dots, G_t = G'$ such that for any $1 \leq i \leq t$, one of G_{i-1}, G_i is obtained from the other by placing a new vertex v at the middle of an edge $\{a, b\}$ (breaking it into two edges) and connecting v to all common neighbors of a and b .*

Along the way, in Proposition 3.1 we show that one can connect any simplicial complex to its barycentric subdivision by a sequence of edge subdivisions (no inverse

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moves are needed). We use this result to rediscover Alexander's result [2, Corollary 10:2d] that in Theorem 1.1 subdivisions at edges suffice; see Corollary 4.1. We explain now an aspect in which our proof is advantageous. In view of Corollary 4.1, one may strengthen Alexander's conjecture that in Theorem 1.1 one can perform all stellar subdivisions before all the inverse stellar subdivisions (see e.g. [9, p. 14, unsolved problem]) as follows:

Problem 1.3. Two simplicial complexes Δ and Δ' are PL-homeomorphic iff they have a common refinement by a sequence of edge subdivisions from each of them.

Our proof of Corollary 4.1 shows that this is true if Δ' is obtained from Δ by some stellar subdivision (while Alexander's proof connects them by a “zigzag” sequence). For further development on Problem 1.3 and its connection to the strong Oda conjecture see the recent [6] and the references therein.

We summarize the above results in the language of graph theory: let X be a simplicial complex, and define a graph $G_s(X) = (V, E)$ as follows. V is the set of simplicial complexes PL-homeomorphic to X , and $\{Y, Z\} \in E$ iff one of Y, Z is obtained from the other by a stellar subdivision, say at a face F . Let $G_e(X)$ be the graph obtained from $G_s(X)$ by deleting the edges for which $|F| > 2$. Let $G_f(X)$ be the graph induced from $G_e(X)$ by restricting to the vertices corresponding to flag complexes. Then $G_f(X) \subseteq G_e(X) \subseteq G_s(X)$ satisfy:

- $G_s(X)$ is connected (Alexander [2]).
- $G_e(X)$ is connected (Alexander [2]).
- $G_f(X)$ is connected (Theorem 5.1).

Next, we apply Theorem 5.1 to problems on f -vectors of flag spheres, and pose two new conjectures about the combinatorial structure forced by their face numbers, analogous to the extremal examples in the upper and lower bound theorems for simplicial spheres. The conjectures are supported by computer experiments – by Theorem 5.1 our algorithm searches through the entire space of flag PL-spheres of any fixed dimension, see Corollary 6.3.

Outline: preliminaries on stellar theory are given in Section 2, barycentric subdivisions are discussed in Section 3, concluding that $G_e(X)$ is connected in Section 4, proof that $G_f(X)$ is connected is given in Section 5, and conjectures for extremal flag spheres are given in Section 6.

2. PRELIMINARIES

For a simplicial complex Δ and a face F in it, let the stellar subdivision of Δ at F be

$$\text{stellar}_\Delta(F) := \{T \in \Delta : F \cap T \neq F\} \cup (\{v_F\} * \partial F * \text{lk}_\Delta(F)),$$

where lk denotes the link, namely

$$\text{lk}_\Delta(F) = \{T \in \Delta : T \cap F = \emptyset, T \cup F \in \Delta\}$$

with $*$ denoting the join product of two simplicial complexes, ∂ the boundary complex of a face, that is, $\partial F = \{T : T \subset F, T \neq F\}$, and v_F is a vertex not in Δ . Consider a geometric realization $||\Delta||$ of Δ . Geometrically, placing the new vertex

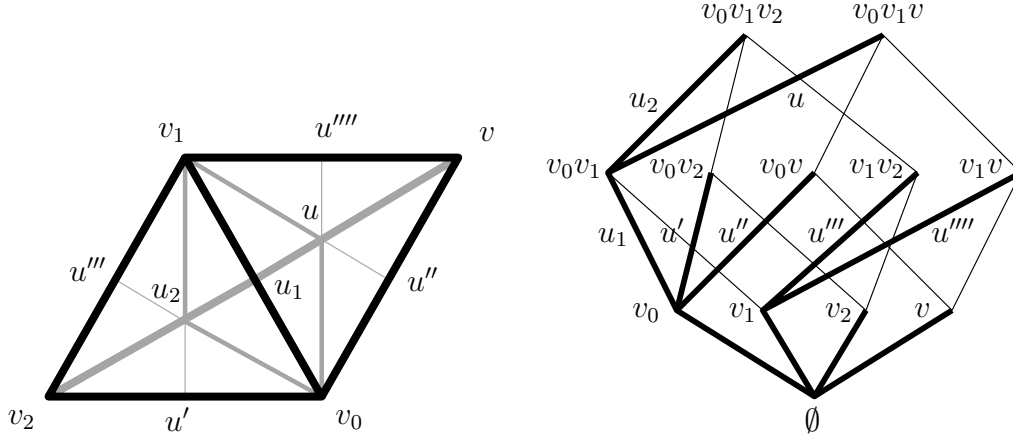


FIGURE 1. Iterated edge subdivisions for two triangles $v_0v_1v_2$ and v_1v_2v according to a spanning tree in the Hasse diagram of the two triangles. The new vertices are inserted in the order $u_1, u_2, u, u', u'', u''', u''''$.

v_F anywhere in the relative interior of $\|F\|$ and taking convex hulls of v_F with the faces of ∂F and the simplices in $\text{lk}_\Delta(F)$ yields the same embedded space for the geometric realization $\|\text{stellar}_\Delta(F)\|$ as $\|\Delta\|$.

Let $\text{br}(\Delta)$ denote the barycentric subdivision of Δ , namely the simplicial complex whose vertices are indexed by the nonempty faces of Δ and whose simplices correspond to a set of faces forming a chain w.r.t. inclusion. To get the same embedded space for the geometric realizations of $\|\Delta\|$ and $\|\text{br}(\Delta)\|$, for each nonempty face $F \in \Delta$ place v_F at the barycenter of $\|F\|$ in the embedding induced by $\|\Delta\|$. It is known that totally ordering the faces of Δ by decreasing dimension and performing stellar subdivisions according to this order changes Δ to $\text{br}(\Delta)$. Geometrically, place each v_F at the barycenter of $\|F\|$.

3. BARYCENTRIC SUBDIVISION: EDGES SUFFICE

Proposition 3.1. *Let Δ be a simplicial complex, and $\text{br}(\Delta)$ denote its barycentric subdivision. Then there is a sequence of edge subdivisions from Δ to $\text{br}(\Delta)$.*

Proof. Choose a maximal chain of simplices in Δ , $\emptyset = \sigma_{-1} \subset \sigma_0 \subset \sigma_1 \subset \sigma_2 \subset \cdots \subset \sigma_t$, with $\dim(\sigma_i) = i$ and $\dim(\Delta) = t$. Denote $v_i = \sigma_i \setminus \sigma_{i-1}$ for $1 \leq i \leq t$ and subdivide the edge σ_1 by a new vertex u_1 . Continue to subdivide edges $\{u_{i-1}, v_i\}$ by a new vertex u_i for $1 < i \leq t$. Now backtrack by replacing σ_t by another (not necessarily inclusion maximal) t -simplex $\sigma = \sigma_{t-1} \cup \{v\}$, if it exists, and subdivide $u_{t-1}v$ by u . Keep the backtracking and edge subdivision process until a (unique) new vertex is added for each simplex in Δ of dimension > 0 .

This process is conveniently described as choosing a spanning tree in the Hasse diagram of the face poset of Δ by a backtracking depth search – the depth of a node equals its rank in the poset, and for pairs $(\emptyset \subseteq \text{vertex})$ the edge subdivision part is empty. (For example, by this rule all edges in Δ containing the fixed vertex σ_0

are subdivided before the other edges in Δ ; this property is not important, as the next claim will show, it just eases the description of the backtracking process.) See Figure 1 for illustration.

We claim that the resulted complex equals $\text{br}(\Delta)$, regardless of the choices made during the backtracking process. This as a special case of the following Claim 3.2. \square

Claim 3.2. Let s be a sequence of stellar subdivisions starting from a simplicial complex K , ending at $s(K)$, and satisfying:

- (i) For any face $F \in K$ with $\dim(F) > 0$ there corresponds a unique vertex of $s(K)$, denoted by v_F , and it is located at the barycenter of $\|F\|$ (note that maybe v_F is added for a stellar subdivision *not* at F , but at a face $G \subseteq \|F\|$ that has been introduced by some earlier subdivision); and
- (ii) if $F_1, F_2 \subseteq F_3$ are three faces in K of dimension > 0 , and if F_1 and F_2 are incomparable, then v_{F_3} does not appear later than both v_{F_1} and v_{F_2} in s .

Then $s(K)$ is combinatorially isomorphic to $\text{br}(K)$.

Proof. First, we reduce the problem to the case where K is a simplex, $K = \overline{T} := \{F : F \subseteq T\}$. For this, let W be a subset of the vertices of K . Then the effect of a stellar subdivision of K at a face F on the induced complex $K[W]$ is nothing if F is not a subset of W and equals $\text{stellar}_{K[W]}(F)$ if $F \subseteq W$. Moreover, the restriction s_W of the sequence s to $K[W]$ satisfies conditions (i) and (ii) in the claim. Thus, by choosing W to be the vertex set of a face in K , we see that the claim will follow if it is true for any simplex \overline{T} .

Assume $K = \overline{T}$ and we prove the claim by induction on $\dim(T)$, where the case $\dim(T) \leq 1$ is trivial. Assume $\dim(T) > 1$. By the induction hypothesis and the remark above on s_W (for all strict subsets of T), we get that the sequence s changes ∂T to $\text{br}(\partial T)$ (note that v_T has no effect on the subdivision of ∂T).

As the geometric realizations of $\text{br}(\overline{T})$ and $s(\overline{T})$ give the same space, it is enough to show that any facet of $s(\overline{T})$ is also a facet of $\text{br}(\overline{T})$. As the restriction of $s(\overline{T})$ to $\|\partial T\|$ is $\text{br}(\partial T)$, it is enough to show that

- (*) for any initial subsequence s' of s that contains v_T , all facets of $s'(\overline{T})$ are of the form $v_T \cup F$ where F is a facet of $s'(\partial T)$.

To prove (*), notice that all vertices v_S that appear before v_T in s correspond to pairwise comparable faces by (ii), hence these faces form a chain of faces in \overline{T} , say with a maximal face T_1 .

Denote by s^F the initial part of s up to vertex v_F , and by $s^F(L)$ the restriction of $s^F(K)$ to $\|L\|$, where L is a subcomplex of K .

By induction on dimension, (*) holds for T_1 , thus all facets in $s^{T_1}(\overline{T}_1)$ are of the form $v_{T_1} \cup F$ where F is a facet of $s^{T_1}(\partial T_1)$. Also, all vertices v_S appearing before v_{T_1} satisfy $S \subseteq T_1$. Hence, all the facets in $s^{T_1}(\overline{T})$ are of the form $v_{T_1} \cup F \cup (T - T_1)$ where F is a facet of $s^{T_1}(\partial T_1)$, thus they contain the face $F_1 = v_{T_1} \cup (T - T_1)$. Note that $\|F_1\|$ contains the barycenter of $\|T\|$ and F_1 is the minimal face of $s^{T_1}(\overline{T})$ with this property. Thus, v_T in s corresponds to a stellar subdivision of $s^{T_1}(\overline{T})$ at F_1 , and the resulted complex $s^T(\overline{T})$ has the property that all its facets have the form

$v_T \cup F'$ where F' is a facet of $s^T(\partial T)$. By (i), any vertex in s that appears after v_T corresponds to a stellar subdivision at a face S contained in $||\partial T||$ and hence all the facets that contain S also contain v_T , thus all facets after the subdivision contain v_T and (*) follows. \square

4. STELLAR THEORY: EDGES SUFFICE

Corollary 4.1. *(Alexander, [2, Corollary 10:2d]) If Δ and Γ are PL-homeomorphic simplicial complexes, then they are connected by a sequence of edge subdivisions and their inverses.*

We give a proof based on Proposition 3.1, whose advantage we explained in the introduction.

Proof. By Theorem 1.1 we can assume that Γ is obtained from Δ by a stellar subdivision at a face F .

Let $s(F)$ be a sequence of edge subdivisions in the simplex \overline{F} , from \overline{F} to $\text{br}(\overline{F})$ as guaranteed by Proposition 3.1. Performing $s(F)$ starting from Δ end in a simplicial complex, denote it Δ' . Let $s(\partial F)$ be a sequence of edge subdivisions in the boundary complex ∂F , from ∂F to $\text{br}(\partial F)$ as guaranteed by Proposition 3.1. Performing $s(\partial F)$ starting from Γ end in a simplicial complex, denote it Γ' .

To finish the proof we show that $\Delta' \cong \Gamma'$ (or equality, with the obvious identifications of vertices given by geometric location at barycenters – which we will use below). Considering the effect of a stellar subdivision on geometric realizations, with each closed face T of the original complex there associated canonically a closed ball consisting of a subcomplex in the result complex, whose underlying space is $||\overline{T}||$. For any face $T \in \Delta$ there is a unique decomposition $T = T_1 \cup F_1$ such that $F_1 \subseteq F$ and $T_1 \cap F = \emptyset$.

Then, as stellar subdivision and join commute, namely for disjoint simplicial complexes A, B and a face $\sigma \in A$, $\text{stellar}_{A*B}(\sigma) = \text{stellar}_A(\sigma) * B$, we get that for $T \in \Delta$ the complex associated with $||\overline{T}||$ in Δ' is $\overline{T}_1 * \text{br}(\overline{F}_1)$. If $F_1 \neq F$, then $T \in \Gamma$ and again $\overline{T}_1 * \text{br}(\overline{F}_1)$ is the corresponding subcomplex in Γ' . If $F_1 = F$, denote by v_F the vertex in the relative interior of $||F||$ in the geometric realizations of both (by the abuse of notation explained above) Δ' and Γ (and Γ'). Then the subcomplex corresponding to $||\overline{T}||$ is as follows: in Δ' it is $\overline{T}_1 * \text{br}(\overline{F}) = \overline{T}_1 * \{v_F\} * \text{br}(\partial F)$; in Γ it is $\overline{T}_1 * \{v_F\} * \partial F$, hence in Γ' it is $\overline{T}_1 * \{v_F\} * \text{br}(\partial F)$. \square

5. FLAG COMPLEXES: EDGES SUFFICE

Recall that a simplicial complex is called *flag* if all the minimal subsets of its vertex set which are not faces have size two; equivalently, if its faces are the cliques in its 1-skeleton.

Theorem 5.1. *Two flag simplicial complexes Δ and Γ are PL-homeomorphic iff they are connected by a sequence of edge subdivisions and their inverses such that all the complexes in the sequence are flag.*

Proof. The ‘if’ part is obvious. As for the ‘only if’ part, by Corollary 4.1, there is a sequence α of simplicial complexes $\Delta = \Delta_0, \Delta_1, \dots, \Delta_t = \Gamma$ such that for each $1 \leq i \leq t$, one of Δ_i and Δ_{i-1} is obtained from the other by an edge subdivision. However, not all complexes in α are flag – we now show how to modify it to a new sequence from Δ to Γ where each Δ_i is flag. The modification is done in steps, where at each step some invariant of the sequence is improved, until a sequence as desired is obtained.

First, we describe the invariant. Define

$$d(\Delta_i) := \sum_{F \notin \Delta_i, \partial F \subseteq \Delta_i, |F| > 2} |F|,$$

thus Δ_i is flag iff $d(\Delta_i) = 0$. (F satisfying $F \notin \Delta_i$ and $\partial F \subseteq \Delta_i$ is called a *missing face* of Δ_i .) For a sequence α as above let $\max(\alpha) := \max_{0 \leq i \leq t} d(\Delta_i)$ and if it is positive let $\text{mult}(\alpha)$ be the number of i ’s for which $d(\Delta_i) = \max(\alpha)$ and define $d(\alpha) := (\max(\alpha), \text{mult}(\alpha))$. Equip \mathbb{N}^2 ($\mathbb{N} = \{1, 2, 3, \dots\}$) with the lexicographic order, namely $(a, b) < (c, d)$ iff either $a < c$ or $a = c$ and $b < d$, and append to it a new element $\hat{0}$, smaller than all, to get a linear order P with a minimum $\hat{0}$. Define $d(\alpha) = \hat{0}$ if $\max(\alpha) = 0$. Thus, α is a sequence as required iff $d(\alpha) = \hat{0}$.

The following observation will be important.

Lemma 5.2. *If Γ' is obtained from a simplicial complex Δ' by an edge subdivision, and that edge is contained in a missing face of Δ' of size > 2 , then $d(\Delta') > d(\Gamma')$.*

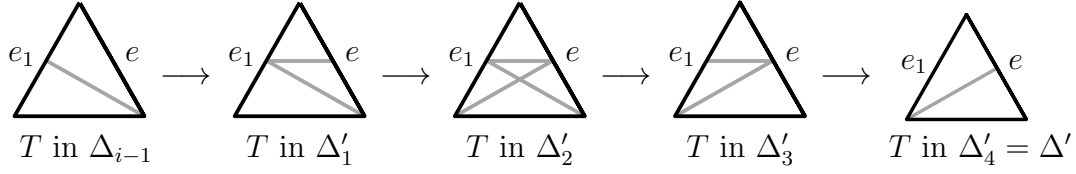
Proof. Let $\{a, b\}$ be the edge subdivided, by a new vertex v . The missing faces of Γ' are obtained from the missing faces of Δ' as follows: if $F'ab := F' \uplus \{a, b\}$ is missing in Δ' replace it by $F'v$ (of smaller size), the other missing faces of Δ' are missing also in Γ' , and the rest of the missing faces of Γ' are of the form vu for some vertex u .

As missing edges don’t effect $d(\cdot)$, and Δ' has a missing face of the form $F'ab$ with F' nonempty, then $d(\Delta') > d(\Gamma')$. \square

The argument above on missing faces also verifies that

Lemma 5.3. *Let Γ' be obtained from a simplicial complex Δ' by an edge subdivision. If the edge subdivided is in no missing face then $d(\Delta') = d(\Gamma')$. In particular, if Δ' is flag then Γ' is flag. \square*

Next, we modify the sequence. Assume $d(\alpha) > \hat{0}$, as else we are done. Call i a *valley* of α ($0 < i < t$) if each of Δ_{i-1} and Δ_{i+1} is obtained from Δ_i by an edge subdivision. As both Δ_0 and Δ_t are flag, by the assumption $d(\alpha) > \hat{0}$ and Lemma 5.3, α has a valley. By Lemma 5.2, α has a valley i such that $d(\Delta_i) = \max(\alpha)$. Consider such i , and let e_1 (resp. e_2) be the edge of Δ_i subdivided to obtain Δ_{i-1} (resp. Δ_{i+1}). As $d(\alpha) > \hat{0}$, there exists a missing face in Δ_i of size > 2 , and let e be an edge contained in it. Denote by Δ' the complex obtained from Δ_i by subdividing at e . In the sequence α replace Δ_i by three consecutive complexes $(\Delta_i, \Delta', \Delta_i)$ to obtain a sequence β ; unless if Δ' equals Δ_{i-1} or Δ_{i+1} where we define (the degenerate case) $\beta = \alpha$, which is treated in the same manner as the non-degenerate case, to be described now. W.l.o.g. assume say $e \neq e_1$.


 FIGURE 2. Sequence of 5 complexes $(\Delta_{i-1}, \Delta'_1, \Delta'_2, \Delta'_3, \Delta'_4 = \Delta')$.

Case 1: e and e_1 are not contained in a common 2-face of Δ_i . Then the two subdivisions, at e and at e_1 , commute (e.g. [2, Corollary 10:2a]). Replace in β the part $(\Delta_{i-1}, \Delta_i, \Delta')$ by the one obtained by commuting the subdivisions, $(\Delta_{i-1}, \Delta'', \Delta')$, and note that $d(\Delta'') \leq d(\Delta') < d(\Delta_i)$ by Lemmas 5.2 and 5.3.

Case 2: otherwise, e and e_1 are in a (unique) 2-face T , and replace in β the part $(\Delta_{i-1}, \Delta_i, \Delta')$ by a sequence of 5 complexes $(\Delta_{i-1}, \Delta'_1, \Delta'_2, \Delta'_3, \Delta'_4 = \Delta')$ as induced by the subdivisions of T described in Figure 2, see also [6, Figure 1A]. Note that $d(\Delta'_j) < d(\Delta_i)$ for $1 \leq j \leq 4$ as each Δ'_j is obtained from Δ_i by a sequence of edge subdivisions that include e .

If $e \neq e_2$, do a similar replacement for the part $(\Delta', \Delta_i, \Delta_{i+1})$ in β ; resulting in a sequence α' , from Δ to Γ . Note that $d(\alpha') < d(\alpha)$.

Thus, after repeating the replacement process finitely many times we arrive at a sequence α'' with $d(\alpha'') = \hat{0}$, as desired. \square

6. APPLICATIONS: FLAG SPHERES

Barnette's lower bound theorem for simplicial polytopes and spheres [3, 4] follows from the inequality on face numbers of the 1-skeleton for all simplicial spheres:

$$g_2 := f_1 - df_0 + \binom{d+1}{2} \geq 0,$$

where $d - 1$ is the dimension of the sphere, and f_i the number of i -dimensional faces. This reduction is known as McMullen–Perles–Walkup reduction (MPW). Barnette proved that equality happens iff the simplicial polytope is *stacked*, and Kalai extended this conclusion to all *homology spheres* [10].

Stronger lower bounds for the case where the homology spheres are *flag* were conjectured in [15, Conjecture 1.4], and a reduction similar to MPW was shown [15, Proposition 3.2] to the following inequality, for all flag homology $(d - 1)$ -spheres (same notation as above):

$$\gamma_2 := f_1 - (2d - 3)f_0 + 2d(d - 2) \geq 0.$$

This inequality is part of Gal's conjecture that the entire γ -vector of flag homology spheres is nonnegative [8]. We conjecture now when equality occurs.

Conjecture 6.1. Let $d \geq 4$ be an integer and Δ be a flag simplicial $(d-1)$ -sphere. Then the following are equivalent:

- (i) $\gamma_2(\Delta) = 0$.
- (ii) There is a sequence of edge contractions from Δ to the boundary of the d -dimensional cross polytope, i.e., to the octahedral $(d-1)$ -sphere, such that all complexes in the sequence are flag spheres.

Moreover, the link of each edge contracted is the octahedral $(d-3)$ -sphere.

We remark that the implication $(ii) \Rightarrow (i)$ is easy, as for edge contraction $K' = K/e$ one has $\gamma_2(K) = \gamma_2(K') + \gamma_1(\text{lk}_K(e))$, and as shown in [8, 13], the only flag spheres for which $\gamma_1 := f_0 - 2d$ vanishes are octahedral. Moreover, for all flag spheres $\gamma_1 \geq 0$ which shows the ‘Moreover’ part in the conjecture.

We now turn to a conjecture on the extremal examples for upper bounds.

Let $T(r, n)$ be the complete r -partite graph on n vertices with the parts as equal size as possible. Turán showed that this graph has more edges than any other graph on n vertices without an $(r+1)$ -clique. The number of i cliques in $T(r, n)$, denoted $f_{i-1}(r, n)$, can be easily computed and is roughly $\binom{r}{i} \left(\frac{n}{r}\right)^i$.

In [16, Conjecture 6.3] it was conjectured that for any flag homology sphere Δ , $\gamma(\Delta)$ is the f -vector of some balanced complex. In particular, from the characterization of such f -vectors [7] it would follow that if Δ is $(d-1)$ -dimensional with n vertices then

$$\gamma_i(\Delta) \leq f_{i-1}(\lfloor \frac{d}{2} \rfloor, n - 2d)$$

for all $2 \leq i \leq \lfloor \frac{d}{2} \rfloor$ (equality for $i = 0, 1$ is clear). What can be said about the case of equality?

Conjecture 6.2. Let $d \geq 4$ be even and Δ be a flag simplicial $(d-1)$ -sphere on n vertices. Then the following are equivalent:

- (i) $\gamma_i(\Delta) = f_{i-1}(\frac{d}{2}, n - 2d)$ for some $2 \leq i \leq \frac{d}{2}$.
- (ii) Δ is the join of $\frac{d}{2}$ cycles of as equal length as possible.

We remark that this conjecture is in big contrast to the usual upper bound theorem for simplicial polytopes (McMullen [12]) and spheres (Stanley [17]), where equality is attained by numerous examples, namely by all neighborly polytopes and spheres. For $d = 4$ it would follow from a conjecture of Gal [8, Conjecture 3.2.2]. Very recently, the case $d = 4$ of the conjecture was confirmed when γ_1 is large enough [1], compare also [1, Conjecture 5.1].

To test the implication $(i) \Rightarrow (ii)$ in Conjecture 6.1, as well as Gal’s conjecture $\gamma_2 \geq 0$, we run the following computer program; here is the algorithm.

- (1) Start with the octahedral d -sphere ($d \geq 3$),
- (2) perform at random either an edge subdivision or a contraction of an edge which is in *no* induced 4-cycle (we call such contraction *admissible*),
- (3) check if $\gamma_2 \geq 0$ and
- (4) if equality occurs, contract edges in *no* induced 4-cycle as long as possible and check if get stuck in the octahedral sphere.
- (5) repeat: go back to (2).

Corollary 6.3. *Fix $d \geq 3$. Our computer program searches exactly through the entire space of d -dimensional flag PL-spheres.*

Proof. First note that the condition on admissible edge contractions guarantees that all the complexes obtained are flag. Indeed, for an admissible contraction of edge $\{a, b\}$ in a flag complex Δ , to a new vertex v , the resulted complex

$$\begin{aligned} \Delta' := & \{T \in \Delta : a, b \notin T\} \\ & \cup \{T \cup \{v\} : T \cap \{a, b\} = \emptyset \text{ and either } T \cup \{a\} \in \Delta \text{ or } T \cup \{b\} \in \Delta\} \end{aligned}$$

satisfies that it has no missing triangles. Suppose by contradiction that $F \in \Delta$, $|F| > 2$ and $F \uplus \{v\}$ is a missing face in Δ' . We'll show that one of a, b is a neighbor of all elements of F in the 1-skeleton of Δ , which implies, as Δ is flag, that $F \uplus \{v\} \in \Delta'$, a contradiction. If b is not a neighbor of some $u' \in F$ then as $\partial(F \uplus \{v\}) \subseteq \Delta'$ we conclude that for any $u' \neq u \in F$, $(F \setminus \{u\}) \cup \{a\} \in \Delta$. As $|F| > 2$ we get that a is a neighbor of all elements of F in Δ ; concluding that Δ' is flag.

In particular, the edges contracted satisfy the link-condition $\text{lk}(\{a, b\}) = \text{lk}(a) \cap \text{lk}(b)$, thus the contractions preserve the PL type of the sphere [14]; clearly the (stellar) edge subdivisions preserve the PL type as well. Note that the inverse of edge subdivision on flag complexes is a special case of an admissible edge contraction. Thus, Theorem 5.1 finishes the proof. \square

Our computer experiments support the above two conjectures.

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